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Complex Perturbed Bernstein-Schurer-Type Operators

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Highlights

- This paper deals with a new generalization of complex perturbed Bernstein-type operators.
- Quantitative upper estimates for simultaneous approximation by the specified operator are obtained.
- Voronovskaya type asymptotic formula and the exact degree of approximation are established in the study.

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Abstract

In the present paper, we describe a new generalization of complex Bernstein-Schurer operators. We attain quantitative upper estimates for the convergence, lower estimates from a qualitative Voronovskaya type result and afterwards establish the exact degree of simultaneous approximation by the specified operator attached to analytical functions in a disk centered at the origin having radius greater than one.

1. INTRODUCTION

In [1], Anastassiou and Gal introduced complex form of the Bernstein-Schurer operators as

$$\mathcal{B}_{\mu,\gamma}(\psi,\omega)=\sum_{\tau=0}^{\mu+\gamma}\widetilde{\Theta}_{\mu,\tau}(\omega)\,\psi\left(\frac{\tau}{\mu}\right), \tag{1}$$

for $\mu\in\mathbb{N}$, non-negative integer γ, $\omega\in\mathbb{C}$, any complex-valued ψ which is analytical in an open disk centered at the origin having radius greater than one, and complex Schurer's fundamental functions $\widetilde{\Theta}_{\mu,\tau}(\omega)$ given by

$$\widetilde{\Theta}_{\mu,\tau}(\omega)=\begin{cases}\binom{\mu+\gamma}{\tau}\omega^{\tau}(1-\omega)^{\mu+\gamma-\tau}, & 0\le\tau\le\mu+\gamma\\ 0, & \tau<0 \text{ or } \tau>\mu+\gamma.\end{cases}$$

The authors attained some quantitative estimates for the uniform convergence, Voronovskaya-type result with quantitative estimate and the exact degree of simultaneous approximation for complex Bernstein-Schurer operators on compact disks. The case $\gamma=0$ produces the classical complex Bernstein operators worked in [2]. It is clear that Schurer's fundamental functions $\widetilde{\Theta}_{\mu,\tau}(\omega)$ satisfy the following

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$$\widetilde{\Theta}_{\mu,\tau}(\omega) = (1-\omega)\widetilde{\Theta}_{\mu-1,\tau}(\omega) + \omega\widetilde{\Theta}_{\mu-1,\tau-1}(\omega), 0 \le \tau \le \mu+\gamma. \tag{2}$$

By (2), complex Bernstein-Schurer operators (1) can be put down as

$$\mathcal{B}_{\mu,\gamma}(\psi,\omega) = \sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(\omega)\left[(1-\omega)\psi\left(\frac{\tau}{\mu}\right) + \omega\psi\left(\frac{\tau+1}{\mu}\right)\right]. \tag{3}$$

Let us remember that divided difference of order τ of a function ψ at $\vartheta_0, \vartheta_1, \dots, \vartheta_\tau$ is represented by $[\vartheta_0, \vartheta_1, \dots, \vartheta_\tau; \psi]$ and $[\vartheta_0; \psi] = \psi(\vartheta_0)$ and

$$[\vartheta_0, \vartheta_1, \dots, \vartheta_\tau; \psi] = \sum_{j=0}^{\tau} \frac{\psi(\vartheta_j)}{\prod_{i=0, i\neq j}^{\tau}(\vartheta_j - \vartheta_i)} \tag{4}$$

(see, p. 184 in [3]). From Lemma 2.1 in [4], we obtain divided difference representation for the operator $\mathcal{B}_{\mu,\gamma}$ as

$$\mathcal{B}_{\mu,\gamma}(\psi,\omega) = \sum_{\tau=0}^{\mu+\gamma-1} \binom{\mu+\gamma-1}{\tau} \frac{\tau!}{\mu^\tau}\left[(1-\omega)[0, \frac{1}{\mu}, \dots, \frac{\tau}{\mu}; \psi] + \omega[\frac{1}{\mu}, \frac{2}{\mu}, \dots, \frac{\tau+1}{\mu}; \psi]\right]\omega^\tau. \tag{5}$$

More recently, Khosravian-Arab et al. [5] described and analyzed a new generalization of Bernstein operators using a novel and simple procedure to enhance the degree of approximation, as shown below:

$$\mathcal{B}^1_\mu(\psi, u) = \sum_{\tau=0}^{\mu} \Theta^1_{\mu,\tau}(u)\,\psi\left(\frac{\tau}{\mu}\right), \qquad u \in [0,1], \tag{6}$$

where $\psi \in C[0,1]$, $\mu \in \mathbb{N}$,

$$\begin{aligned} &\Theta^1_{\mu,\tau}(u) = \mathcal{D}(u,\mu)\Theta_{\mu-1,\tau}(u) + \mathcal{D}(1-u,\mu)\Theta_{\mu-1,\tau-1}(u), \qquad 1 \le \tau \le \mu-1 \\ &\Theta^1_{\mu,0}(u) = \mathcal{D}(u,\mu)(1-u)^{\mu-1}, \qquad \Theta^1_{\mu,\mu}(u) = \mathcal{D}(1-u,\mu)u^{\mu-1}, \end{aligned} \tag{7}$$

in which $\Theta_{\mu,\tau}(u) = \binom{\mu}{\tau}u^\tau(1-u)^{\mu-\tau}$ and

$$\mathcal{D}(u,\mu) = \mathcal{D}_1(\mu)u + \mathcal{D}_0(\mu), \ \ \mu = 0,1,\dots,$$

with two unknown real sequences $\mathcal{D}_1(\mu)$ and $\mathcal{D}_0(\mu)$. We note that for $\mathcal{D}_1(\mu) = -1$ and $\mathcal{D}_0(\mu) = 1$, the operator $\mathcal{B}^1_\mu$ yields the Bernstein operator. The operator $\mathcal{B}^1_\mu$ is named as perturbed Bernstein-type operator. We may refer to a few works on perturbed Bernstein operators and some generalizations of these operators in the complex and real situations (see e.g. [6-12]).

In the present paper, stimulating from above studies related to the complex Bernstein-Schurer operators and perturbed Bernstein operators of real variable, we present a new generalized complex Bernstein-Schurer operators as

$$\mathcal{B}^1_{\mu,\gamma}(\psi,\omega) = \sum_{\tau=0}^{\mu+\gamma} \widetilde{\Theta}^1_{\mu,\tau}(\omega)\,\psi\left(\frac{\tau}{\mu}\right), \tag{8}$$

where $\mu \in \mathbb{N}$, non-negative integer γ, $\omega \in \mathbb{C}$, any complex-valued ψ which is analytical in an open disk centered at the origin having a radius greater than one, and

$$\widetilde{\Theta}^1_{\mu,\tau}(\omega) = \mathcal{D}(\omega,\mu)\widetilde{\Theta}_{\mu-1,\tau}(\omega) + \mathcal{D}(1-\omega,\mu)\widetilde{\Theta}_{\mu-1,\tau-1}(\omega), \qquad 1 \le \tau \le \mu+\gamma-1$$
$$\widetilde{\Theta}^1_{\mu,0}(\omega) = \mathcal{D}(\omega,\mu)(1-\omega)^{\mu+\gamma-1}, \qquad \widetilde{\Theta}^1_{\mu,\mu+\gamma}(\omega) = \mathcal{D}(1-\omega,\mu)\omega^{\mu+\gamma-1}, \tag{9}$$

in which $\widetilde{\Theta}_{\mu,\tau}(z)$ is complex Schurer's fundamental function and

$$\mathcal{D}(\omega,\mu) = \mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu), \;\; \mu = 0,1,\ldots,$$

with two unidentified real sequences $\mathcal{D}_1(\mu)$ and $\mathcal{D}_0(\mu)$. For these real sequences, we are concerned with the next situations:

(1) If $\mathcal{D}_1(\mu) = 0$, then $\mathcal{D}_0(\mu) = \frac{1}{2}$.
(2) If $0 < \mathcal{D}_1(\mu) < 1$, then $\mathcal{D}_0(\mu) > 0$.
(3) If $\mathcal{D}_1(\mu) = 1$, then $\mathcal{D}_0(\mu) = 0$.
(4) If $-1 < \mathcal{D}_1(\mu) < 0$, then $\mathcal{D}_0(\mu) > 0$ and $\mathcal{D}_1(\mu) + \mathcal{D}_0(\mu) > 0$.
(5) If $\mathcal{D}_1(\mu) = -1$, then $\mathcal{D}_0(\mu) = 1$.

We called the operator $\mathcal{B}^1_{\mu,\gamma}$ as complex perturbed Bernstein-Schurer operators. When $\mathcal{D}_1(\mu) = -1$ and $\mathcal{D}_0(\mu) = 1$ with $\gamma = 0$, the operator $\mathcal{B}^1_{\mu,\gamma}$ gives the classical complex Bernstein operator investigated thoroughly in [2]. If we choose $\mathcal{D}_1(\mu) = -1$ and $\mathcal{D}_0(\mu) = 1$, the operator $\mathcal{B}^1_{\mu,\gamma}$ becomes complex Bernstein-Schurer operators analyzed in [1]. Furthermore, in the case $\gamma = 0$ the operator $\mathcal{B}^1_{\mu,\gamma}$ reduces to the complex perturbed Bernstein-type operators examined in [9].

Using (9), complex perturbed Bernstein-Schurer operator given by (8) can be expressed as

$$\mathcal{B}^1_{\mu,\gamma}(\psi,\omega) = \sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(\omega)\left[\mathcal{D}(\omega,\mu)\psi\left(\frac{\tau}{\mu}\right) + \mathcal{D}(1-\omega,\mu)\psi\left(\frac{\tau+1}{\mu}\right)\right], \tag{10}$$

from which we have the analogous statement of the complex Bernstein-Schurer operators in (3) when $\mathcal{D}_1(\mu) = -1$ and $\mathcal{D}_0(\mu) = 1$.

Likewise, we can state the following representation for the operator $\mathcal{B}^1_{\mu,\gamma}$ via divided difference

$$\mathcal{B}^1_{\mu,\gamma}(\psi,\omega) = \sum_{\tau=0}^{\mu+\gamma-1} \binom{\mu+\gamma-1}{\tau} \frac{\tau!}{\mu^\tau}\left[\mathcal{D}(\omega,\mu)[0,\frac{1}{\mu},\ldots,\frac{\tau}{\mu};\psi] + \mathcal{D}(1-\omega,\mu)[\frac{1}{\mu},\frac{2}{\mu},\ldots,\frac{\tau+1}{\mu};\psi]\right]\omega^\tau. \tag{11}$$

Here, first of all we attain quantitative upper estimates for the operator $\mathcal{B}^1_{\mu,\gamma}$ and its derivatives on compact disks. Subsequently, we attain lower estimates by using a qualitative Voronovskaya type result. In conclusion, we set up the exact degree of simultaneous approximation for the newly defined operator $\mathcal{B}^1_{\mu,\gamma}$.

2. AUXILIARY RESULTS

Before presenting main results, we recall some symbols that will be used throughout the paper.

Let $H_{\mathcal{R}}$, $\mathcal{R} > 1$, indicate the open disk $H_{\mathcal{R}} := \{\omega \in \mathbb{C}: |\omega| < \mathcal{R}\}$ and the closed disk involved in $H_{\mathcal{R}}$, $\overline{H}_\rho := \{\omega \in \mathbb{C}: |\omega| \le \rho, 1 \le \rho < \mathcal{R}\}$ and moreover $\|\psi\|_\rho = max\{|\psi(\omega)|: |\omega| \le \rho\}$.

We shall use notation $e_s(\omega) = \omega^s, s \in \mathbb{N} \cup \{0\}, \omega \in \mathbb{C}$ and presume that $\mathcal{B}^1_{\mu,\gamma}(e_0,\omega) = 1$, in other words

$$2\mathcal{D}_0(\mu) + \mathcal{D}_1(\mu) = 1. \tag{12}$$

Evidently, we say that the foregoing sequences $\mathcal{D}_i(\mu), i = 0,1$, are bounded, that is $|\mathcal{D}_i(\mu)| \leq \mathcal{N}, \mathcal{N} > 0$. Before evidencing the principal results, we require the subsequent consequences.

Lemma 2.1. For the real operator $\mathcal{B}^1_{\mu,\gamma}$ given by (10), the moments are obtained as

$$\mathcal{B}^1_{\mu,\gamma}(e_0,u) = 2\mathcal{D}_0(\mu) + \mathcal{D}_1(\mu) = 1,$$

$$\mathcal{B}^1_{\mu,\gamma}(e_1,u) = \frac{\big(\mu+\gamma-1-\mathcal{D}_1(\mu)\big)u + \mathcal{D}_1(\mu) + \mathcal{D}_0(\mu)}{\mu},$$

$$\mathcal{B}^1_{\mu,\gamma}(e_2,u) = \frac{1}{\mu^2}\big\{(\mu+\gamma-1)\big(\mu+\gamma-2-2\mathcal{D}_1(\mu)\big)u^2 + \big[(\mu+\gamma-1)\big(1+2\mathcal{D}_1(\mu)+2\mathcal{D}_0(\mu)\big)$$

$$-\mathcal{D}_1(\mu)\big]u + \mathcal{D}_1(\mu) + \mathcal{D}_0(\mu)\big\}.$$

Theorem 2.1. If $\psi \in C[0,1+\gamma]$, then

$$\lim_{\mu\to\infty} \mathcal{B}^1_{\mu,\gamma}(\psi,u) = \psi(u)$$

uniformly on $[0,1+\gamma]$.

Proof. Because $\mathcal{D}_0(\mu)$ and $\mathcal{D}_1(\mu)$ verify the cases (1)-(5), the operator $\mathcal{B}^1_{\mu,\gamma}$ given by (10) in the real case is positive. Taking into consideration that the sequences $\mathcal{D}_0(\mu), \mathcal{D}_1(\mu)$ are bounded, by making use of Korovkin theorem and Lemma 2.1, we have the uniform convergence of the operator $\mathcal{B}^1_{\mu,\gamma}$.

Theorem 2.2. Assume that $\mathcal{D}_i(\mu), i = 0,1$, are convergent sequences and $\mathcal{L}_1 = \lim_{\mu\to\infty} \mathcal{D}_1(\mu)$. If $\psi \in C^2[0,1+\gamma]$ and $u \in [0,1]$, then

$$\lim_{\mu\to\infty} \mu\big[\mathcal{B}^1_{\mu,\gamma}(\psi,u) - \psi(u)\big] = \left[\gamma u + \frac{(1-2u)}{2}(1+\mathcal{L}_1)\right]\psi'(u) + \frac{u(1-u)}{2}\psi''(u).$$

Proof. We write down

$$\mu\big[\mathcal{B}^1_{\mu,\gamma}(\psi,u) - \psi(u)\big] = \mu\big[\mathcal{B}_{\mu,\gamma}(\psi,u) - \psi(u)\big] + \mu\big[\mathcal{B}^1_{\mu,\gamma}(\psi,u) - \mathcal{B}_{\mu,\gamma}(\psi,u)\big] \tag{13}$$

$$=: G_1(u) + G_2(u).$$

Note that the limit of $G_1(u)$ is familiar from Theorem 3.3 in [13] as a special case, namely, $\gamma u\psi'(u) + \frac{u(1-u)}{2}\psi''(u)$. Furthermore,

$$G_2(u) = \mu\left\{\sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(u)\,[\mathcal{D}_1(\mu)u + \mathcal{D}_0(\mu) - 1 + u]\,\psi\left(\frac{\tau}{\mu}\right)\right.$$

$$\left. + \sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(u)\,[\mathcal{D}_1(\mu) - \mathcal{D}_1(\mu)u + \mathcal{D}_0(\mu) - u]\psi\left(\frac{\tau+1}{\mu}\right)\right\}$$

$$= \mu\left\{\sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(u)\,[(1+\mathcal{D}_1(\mu))u + \mathcal{D}_0(\mu) - (2\mathcal{D}_0(\mu) + \mathcal{D}_1(\mu))]\,\psi\left(\frac{\tau}{\mu}\right)\right.$$

$$+\sum_{\tau=0}^{\mu+\gamma-1}\widetilde{\Theta}_{\mu-1,\tau}(u)\,[-(1+\mathcal{D}_1(\mu))u+\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)]\psi\left(\frac{\tau+1}{\mu}\right)\Bigg\}$$

$$=-\mu\sum_{\tau=0}^{\mu+\gamma-1}\widetilde{\Theta}_{\mu-1,\tau}(u)\,[-(1+\mathcal{D}_1(\mu))u+\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)]\psi\left(\frac{\tau}{\mu}\right)$$

$$+\mu\sum_{\tau=0}^{\mu+\gamma-1}\widetilde{\Theta}_{\mu-1,\tau}(u)\,[-(1+\mathcal{D}_1(\mu))u+\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)]\psi\left(\frac{\tau+1}{\mu}\right)$$

$$=[-(1+\mathcal{D}_1(\mu))u+\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)]\mu\sum_{\tau=0}^{\mu+\gamma-1}\widetilde{\Theta}_{\mu-1,\tau}(u)\left[\psi\left(\frac{\tau+1}{\mu}\right)-\psi\left(\frac{\tau}{\mu}\right)\right]\Bigg\}$$

$$=[-(1+\mathcal{D}_1(\mu))u+\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)]\frac{\mu}{\mu+\gamma}\left(\mathcal{B}_{\mu,\gamma}\psi\right)'(u), \tag{14}$$

where the operator $\mathcal{B}_{\mu,\gamma}$ is Bernstein-Schurer operator. From [14], we have $\lim_{\mu\to\infty}\left(\mathcal{B}_{\mu,\gamma}\psi\right)'(u)=\psi'(u)$. By (12) and (14), we obtain

$$\lim_{\mu\to\infty}G_2(u)=\left[-(1+\mathcal{L}_1)u+\frac{1+\mathcal{L}_1}{2}\right]\psi'(u).$$

Thus, using $\lim_{\mu\to\infty}\mu\left[\mathcal{B}_{\mu,\gamma}(\psi,u)-\psi(u)\right]=\gamma u\psi'(u)+\frac{u(1-u)}{2}\psi''(u)$, the theorem is proved.

3. PRINCIPAL RESULTS

The preliminary result is given by the relationship below.

Theorem 3.1. Assume that ρ and $\mathcal{R}$ are constants such that $1\le\rho<\mathcal{R}$, $\gamma\in\mathbb{N}\cup\{0\}$ and ψ is analytical in $H_{\mathcal{R}}$, with $\psi(\omega)=\sum_{\tau=0}^{\infty}c_\tau\omega^\tau$. Then, for all $\omega\in\overline{H}_\rho$ and $\mu\in\mathbb{N}$, we get

$$\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)=\sum_{\tau=0}^{\infty}c_\tau\,\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega).$$

Proof. For any $\upsilon\in\mathbb{N}$, we possess $\psi_\upsilon(\omega)=\sum_{\tau=0}^{\upsilon}c_\tau\omega^\tau$, $|\omega|\le\rho$. Since $\mathcal{B}^1_{\mu,\gamma}$ is linear, we can obviously set down

$$\mathcal{B}^1_{\mu,\gamma}(\psi_\upsilon,\omega)=\sum_{\tau=0}^{\upsilon}c_\tau\,\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega),$$

for all $|\omega|\le\rho,\rho\ge1$. Hence, it remains to confirm that $\lim_{\upsilon\to\infty}\mathcal{B}^1_{\mu,\gamma}(\psi_\upsilon,\omega)=\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)$ for any fixed $\mu\in\mathbb{N}$ and $|\omega|\le\rho$. We claim that for the sequences $\mathcal{D}_0(\mu)$ and $\mathcal{D}_1(\mu)$, by the possible cases (1)-(5) we find $\mathcal{D}_0(\mu)\ge0$ and $\mathcal{D}_0(\mu)+\mathcal{D}_1(\mu)\ge0$. For all $|\omega|\le\rho$ with $\rho\ge1$, from (10), we get

$$\left|\mathcal{B}^1_{\mu,\gamma}(\psi_\upsilon,\omega)-\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)\right|$$

$$\leq \left| \sum_{\tau=0}^{\mu+\gamma-1} \widetilde{\Theta}_{\mu-1,\tau}(\omega) \left[(\mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu))(\psi_\upsilon - \psi)\left(\frac{\tau}{\mu}\right) + (\mathcal{D}_1(\mu)(1-\omega) + \mathcal{D}_0(\mu))(\psi_\upsilon - \psi)\left(\frac{\tau+1}{\mu}\right) \right] \right|$$

$$\leq \sum_{\tau=0}^{\mu+\gamma-1} \left|\widetilde{\Theta}_{\mu-1,\tau}(\omega)\right| \left[(|\mathcal{D}_1(\mu)\omega| + |\mathcal{D}_0(\mu)|) \left|(\psi_\upsilon - \psi)\left(\frac{\tau}{\mu}\right)\right| \right.$$

$$\left. +(|\mathcal{D}_1(\mu) + \mathcal{D}_0(\mu)| + |\mathcal{D}_1(\mu)\omega|) \left|(\psi_\upsilon - \psi)\left(\frac{\tau+1}{\mu}\right)\right| \right]$$

$$= \sum_{\tau=0}^{\mu+\gamma-1} \binom{\mu+\gamma-1}{\tau} |\omega^\tau||1-\omega|^{\mu+\gamma-1-\tau} \left[(|\mathcal{D}_1(\mu)\omega| + \mathcal{D}_0(\mu)) \left|(\psi_\upsilon - \psi)\left(\frac{\tau}{\mu}\right)\right| \right.$$

$$\left. +(\mathcal{D}_1(\mu) + \mathcal{D}_0(\mu) + |\mathcal{D}_1(\mu)\omega|) \left|(\psi_\upsilon - \psi)\left(\frac{\tau+1}{\mu}\right)\right| \right]$$

$$\leq \sum_{\tau=0}^{\mu+\gamma-1} \binom{\mu+\gamma-1}{\tau} \rho^\tau (1+\rho)^{\mu+\gamma-1-\tau} \|\psi_\upsilon - \psi\|_\rho [|\mathcal{D}_1(\mu)|\rho + \mathcal{D}_0(\mu) + \mathcal{D}_1(\mu) + \mathcal{D}_0(\mu) + |\mathcal{D}_1(\mu)|\rho]$$

$$\leq \sum_{\tau=0}^{\mu+\gamma-1} \binom{\mu+\gamma-1}{\tau} (1+\rho)^{\mu+\gamma-1} \|\psi_\upsilon - \psi\|_\rho [2|\mathcal{D}_1(\mu)|\rho + 2\mathcal{D}_0(\mu) + \mathcal{D}_1(\mu)]$$

$$\leq (1+\rho)^{\mu+\gamma-1} 2^{\mu+\gamma-1} \|\psi_\upsilon - \psi\|_\rho [2|\mathcal{D}_1(\mu)|\rho + \rho]$$

$$\leq (2+2\rho)^{\mu+\gamma-1} (2\mathcal{N}+1)\rho \|\psi_\upsilon - \psi\|_\rho,$$

for $|\mathcal{D}_1(\mu)| \leq \mathcal{N}, \mathcal{N} > 0$. Since $\sum_{\tau=0}^{\infty} c_\tau \omega^\tau$ is absolutely and uniformly convergent on each compact subset of $H_{\mathcal{R}}$, we obtain $\lim_{\upsilon\to\infty} \|\psi_\upsilon - \psi\|_\rho = 0$ for $|\omega| \leq \rho$ with $\rho \geq 1$. This finishes the proof.

Quantitative upper estimates concerning simultaneous approximation are presented as below.

Theorem 3.2. Assume that $\psi: H_{\mathcal{R}} \to \mathbb{C}$ is analytical in $H_{\mathcal{R}}$, for fixed $\gamma \in \mathbb{N} \cup \{0\}$ and $\mathcal{R} > \gamma + 1$ with $\psi(\omega) = \sum_{\tau=0}^{\infty} c_\tau \omega^\tau$.

i) Let $1 \leq \rho < \rho(1+\gamma) < \mathcal{R}$ be arbitrary fixed. Then, for all $|\omega| \leq \rho$, one has

$$\left|\mathcal{B}^1_{\mu,\gamma}(\psi,\omega) - \psi(\omega)\right| \leq \frac{M^1_{\rho,\gamma}(\psi)}{\mu},$$

where

$$M^1_{\rho,\gamma}(\psi) = 3(1+\mathcal{N})[\rho(1+\gamma)]^2 \sum_{\tau=2}^{\infty} |c_\tau|\tau(\tau-1)\,[\rho(1+\gamma)]^{\tau-2}$$

$$+2\rho(1+\gamma) \sum_{\tau=1}^{\infty} |c_\tau|\tau\,[\rho(1+\gamma)]^{\tau-1} + \sum_{\tau=0}^{\infty} |c_\tau|\,[\rho(1+\gamma)]^\tau - \sum_{\tau=0}^{\infty} |c_\tau|\,\rho^\tau < \infty.$$

$ii)$ If $1 \leq \rho < \rho_1 < \rho_1(1+\gamma) < \mathcal{R}$, for all $|\omega| \leq \rho$ and $\mu, s \in \mathbb{N}$, one has

$$\left|\left(\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)\right)^{(s)} - \psi^{(s)}(\omega)\right| \leq \frac{M^1_{\rho_1,\gamma}(\psi)\, s!\, \rho_1}{\mu(\rho_1-\rho)^{s+1}},$$

where $M^1_{\rho_1,\gamma}(\psi)$ is defined as in (i).

Proof. $i)$ From Theorem 3.1, for all $|\omega| \leq \rho$ we make out

$$\left|\mathcal{B}^1_{\mu,\gamma}(\psi,\omega) - \psi(\omega)\right| \leq \sum_{\tau=0}^{\infty} |c_\tau| \left|\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega) - e_\tau(\omega)\right|.$$

Right now, we will find out the upper bound for $\left|\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega) - e_\tau(\omega)\right|$. We observe two cases: $0 \leq \tau \leq \mu+\gamma-1$ and $\tau > \mu+\gamma-1$.

Case 1: Because $\mathcal{B}^1_{\mu,\gamma}(e_0,\omega) - e_0(\omega) = 0$, we examine $1 \leq \tau \leq \mu+\gamma-1$. By (11), indicating

$$T^{\gamma,1}_{\mu,v,\tau} \coloneqq \binom{\mu+\gamma-1}{v} \frac{v!}{\mu^v} \left[0, \frac{1}{\mu}, \dots, \frac{v}{\mu}; e_\tau\right] \tag{15}$$

and

$$T^{\gamma,1}_{\mu,1+v,\tau} \coloneqq \binom{\mu+\gamma-1}{v} \frac{v!}{\mu^v} \left[\frac{1}{\mu}, \frac{2}{\mu}, \dots, \frac{v+1}{\mu}; e_\tau\right] \tag{16}$$

for $0 \leq v \leq \mu+\gamma-1, \mu \in \mathbb{N}, \tau \in \mathbb{N} \cup \{0\}$, using the trait of divided difference formula we obtain that

$$T^{\gamma,1}_{\mu,\tau,\tau} = T^{\gamma,1}_{\mu,1+\tau,\tau} = \binom{\mu+\gamma-1}{v} \frac{v!}{\mu^v} = \frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^\tau}. \tag{17}$$

Consequently, from (11) we have

$$\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega) = \sum_{v=0}^{\mu+\gamma-1} \left\{\left(\mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v,\tau} + \left(\mathcal{D}_1(\mu)(1-\omega) + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v+1,\tau}\right\} \omega^v. \tag{18}$$

Making use of convexity of all order of e_τ, $T^{\gamma,1}_{\mu,v,\tau}$ and $T^{\gamma,1}_{\mu,v+1,\tau} \geq 0$. From (15), (16) and (18), for any $|\omega| \leq \rho, 1 \leq \rho < \mathcal{R}$, we attain

$$\left|\mathcal{B}^1_{\mu,\gamma}(e_\tau,\omega) - e_\tau(\omega)\right|$$

$$= \left|\sum_{v=0}^{\tau} \left\{\left(\mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v,\tau} + \left(\mathcal{D}_1(\mu)(1-\omega) + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v+1,\tau}\right\} e_v(\omega) - e_\tau(\omega)\right|$$

$$= \left|\left\{\left(\mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,\tau,\tau} + \left(\mathcal{D}_1(\mu)(1-\omega) + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,\tau+1,\tau} - 1\right\} e_\tau(\omega)\right.$$

$$\left. + \sum_{v=0}^{\tau-1} \left\{\left(\mathcal{D}_1(\mu)\omega + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v,\tau} + \left(\mathcal{D}_1(\mu)(1-\omega) + \mathcal{D}_0(\mu)\right) T^{\gamma,1}_{\mu,v+1,\tau}\right\} e_v(\omega)\right|$$

$$\leq \left|\left\{\left(2\mathcal{D}_0(\mu) + \mathcal{D}_1(\mu)\right) \frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^\tau} - 1\right\} e_\tau(\omega)\right|$$

$$+\sum_{v=0}^{\tau-1}\left\{|\mathcal{D}_1(\mu)\omega+\mathcal{D}_0(\mu)|T_{\mu,v,\tau}^{\gamma,1}+|\mathcal{D}_1(\mu)-\mathcal{D}_1(\mu)\omega+\mathcal{D}_0(\mu)|T_{\mu,v+1,\tau}^{\gamma,1}\right\}|e_v(\omega)|$$

$$\leq\left|\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^\tau}-1\right|\rho^\tau$$

$$+(|\mathcal{D}_1(\mu)|\rho+\mathcal{D}_0(\mu))\rho^{\tau-1}\sum_{v=0}^{\tau-1}T_{\mu,v,\tau}^{\gamma,1}+(|\mathcal{D}_1(\mu)|\rho+\mathcal{D}_1(\mu)+\mathcal{D}_0(\mu))\rho^{\tau-1}\sum_{v=0}^{\tau-1}T_{\mu,v+1,\tau}^{\gamma,1}$$

$$\leq\rho^\tau\left|\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^\tau}-1\right|+(|\mathcal{D}_1(\mu)|+\mathcal{D}_0(\mu))\rho^\tau\left[\sum_{v=0}^{\tau}T_{\mu,v,\tau}^{\gamma,1}-T_{\mu,\tau,\tau}^{\gamma,1}\right]$$

$$+(|\mathcal{D}_1(\mu)|+\mathcal{D}_1(\mu)+\mathcal{D}_0(\mu))\rho^\tau\left[\sum_{v=0}^{\tau}T_{\mu,v+1,\tau}^{\gamma,1}-T_{\mu,\tau+1,\tau}^{\gamma,1}\right]$$

$$\leq\rho^\tau\left|\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^\tau}-1\right|+(|\mathcal{D}_1(\mu)|+\mathcal{D}_0(\mu))$$

$$\times\left[\sum_{v=0}^{\mu+\gamma-1}\binom{\mu+\gamma-1}{v}\frac{v!}{\mu^v}\left[0,\frac{1}{\mu},\dots,\frac{v}{\mu};e_\tau\right]-T_{\mu,\tau,\tau}^{\gamma,1}\right]\rho^\tau$$

$$+\left(|\mathcal{D}_1(\mu)|+\mathcal{D}_1(\mu)+\mathcal{D}_0(\mu)\right)\left[\sum_{v=0}^{\mu+\gamma-1}\binom{\mu+\gamma-1}{v}\frac{v!}{\mu^v}\left[\frac{1}{\mu},\frac{2}{\mu},\dots,\frac{v+1}{\mu};e_\tau\right]-T_{\mu,\tau+1,\tau}^{\gamma,1}\right]\rho^\tau. \tag{19}$$

Since $\tau\leq\mu+\gamma$, we obtain

$$\sum_{v=0}^{\mu+\gamma-1}\binom{\mu+\gamma-1}{v}\frac{v!}{\mu^v}\left[0,\frac{1}{\mu},\dots,\frac{v}{\mu};e_\tau\right]\leq\sum_{v=0}^{\mu+\gamma}\binom{\mu+\gamma-1}{v}\frac{v!}{\mu^v}\left[0,\frac{1}{\mu},\dots,\frac{v}{\mu};e_\tau\right]$$

$$\leq\sum_{v=0}^{\mu+\gamma}\binom{\mu+\gamma}{v}\frac{v!}{\mu^v}\left[0,\frac{1}{\mu},\dots,\frac{v}{\mu};e_\tau\right]$$

$$=\mathcal{B}_{\mu,\gamma}(e_\tau,1)=\left(\frac{\mu+\gamma}{\mu}\right)^\tau. \tag{20}$$

Moreover, via (5) we conclude that

$$\sum_{v=0}^{\mu+\gamma-1}\binom{\mu+\gamma-1}{v}\frac{v!}{\mu^v}\left[\frac{1}{\mu},\frac{2}{\mu},\dots,\frac{v+1}{\mu};e_\tau\right]=\mathcal{B}_{\mu,\gamma}(e_\tau,1)=\left(\frac{\mu+\gamma}{\mu}\right)^\tau. \tag{21}$$

Hence, from (20) and (21), by simple calculations (19) follows that

$$\left|\mathcal{B}_{\mu,\gamma}^1(e_\tau,\omega)-e_\tau(\omega)\right|$$

$$\leq \rho^{\tau}\left|\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}-1\right|+(|\mathcal{D}_1(\mu)|+\mathcal{D}_0(\mu))\left[\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-T_{\mu,\tau,\tau}^{\gamma,1}\right]\rho^{\tau}$$

$$+\left(|\mathcal{D}_1(\mu)|+\mathcal{D}_1(\mu)+\mathcal{D}_0(\mu)\right)\left[\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-T_{\mu,\tau+1,\tau}^{\gamma,1}\right]\rho^{\tau}$$

$$\leq \rho^{\tau}\left|\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}-1\right|$$

$$+(1+2|\mathcal{D}_1(\mu)|)\rho^{\tau}\left[\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}\right]$$

$$\leq \rho^{\tau}\left|\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}\right|+\rho^{\tau}\left|\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-1\right|$$

$$+(1+2|\mathcal{D}_1(\mu)|)\rho^{\tau}\left[\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}\right]$$

$$=2(1+|\mathcal{D}_1(\mu)|)\rho^{\tau}\left|\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}\right|+\rho^{\tau}\left|\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-1\right|. \qquad (22)$$

According to the following inequality

$$1-\prod_{m=1}^{\tau}\vartheta_m\leq\sum_{m=1}^{\tau}(1-\vartheta_m),\ \ 0\leq\vartheta_m\leq 1,\ \ m=1,\dots,\tau,$$

(see, p. 69 in [2]), we may write down

$$\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{\mu^{\tau}}$$

$$=\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}\left[1-\frac{(\mu+\gamma-1)(\mu+\gamma-2)\dots(\mu+\gamma-\tau)}{(\mu+\gamma)^{\tau}}\right]$$

$$=\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}\left[1-\prod_{m=1}^{\tau}\frac{\mu+\gamma-m}{\mu+\gamma}\right]$$

$$\leq\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}\sum_{m=1}^{\tau}\left(1-\frac{\mu+\gamma-m}{\mu+\gamma}\right)$$

$$=\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}\sum_{m=1}^{\tau}\frac{m}{\mu+\gamma}=\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}\frac{\tau(\tau+1)}{2(\mu+\gamma)}=\left(1+\frac{\gamma}{\mu}\right)^{\tau}\frac{\tau(\tau-1)+2\tau}{2(\mu+\gamma)}$$

$$\leq(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{2(\mu+\gamma)}. \qquad (23)$$

On the other hand,

$$\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}-1=\frac{(\mu+\gamma)^{\tau}-\mu^{\tau}}{\mu^{\tau}}$$

$$=\frac{\mu^{\tau}+\binom{\tau}{1}\mu^{\tau-1}\gamma+\binom{\tau}{2}\mu^{\tau-2}\gamma^{2}+\cdots+\gamma^{\tau}-\mu^{\tau}}{\mu^{\tau}}$$

$$=\frac{1}{\mu^{\tau}}\sum_{m=0}^{\tau-1}\binom{\tau}{m}\mu^{m}\gamma^{\tau-m}. \tag{24}$$

Replacing (23) and (24) into (22), we find

$$\left|\mathcal{B}_{\mu,\gamma}^{1}(e_{\tau},\omega)-e_{\tau}(\omega)\right|\leq 2(1+|\mathcal{D}_{1}(\mu)|)\rho^{\tau}(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{2(\mu+\gamma)}+\frac{\rho^{\tau}}{\mu^{\tau}}\sum_{m=0}^{\tau-1}\binom{\tau}{m}\mu^{m}\gamma^{\tau-m}$$

$$\leq(1+|\mathcal{D}_{1}(\mu)|)\rho^{\tau}(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{\mu+\gamma}+\frac{\rho^{\tau}}{\mu^{\tau}}\mu^{\tau-1}\left\{\sum_{m=0}^{\tau}\binom{\tau}{m}\gamma^{\tau-m}-1\right\}$$

$$\leq(1+|\mathcal{D}_{1}(\mu)|)\rho^{\tau}(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{\mu+\gamma}+\frac{\rho^{\tau}}{\mu}[(1+\gamma)^{\tau}-1].$$

Because $\mathcal{D}_{1}(\mu)$ is bounded, the last inequality reduces to

$$\left|\mathcal{B}_{\mu,\gamma}^{1}(e_{\tau},\omega)-e_{\tau}(\omega)\right|\leq(1+\mathcal{N})\rho^{\tau}(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{\mu+\gamma}+\frac{\rho^{\tau}}{\mu}[(1+\gamma)^{\tau}-1]$$

$$\leq(1+\mathcal{N})\rho^{\tau}(1+\gamma)^{\tau}\frac{\tau(\tau-1)+2\tau}{\mu}+\frac{\rho^{\tau}}{\mu}[(1+\gamma)^{\tau}-1].$$

Case 2: For $\tau>\mu+\gamma-1\geq 1$ and $|\omega|\leq\rho$ with $1\leq\rho<\mathcal{R}$, we attain

$$\left|\mathcal{B}_{\mu,\gamma}^{1}(e_{\tau},\omega)-e_{\tau}(\omega)\right|$$

$$\leq\left|\sum_{v=0}^{\mu+\gamma-1}\left\{\left(\mathcal{D}_{1}(\mu)\omega+\mathcal{D}_{0}(\mu)\right)T_{\mu,v,\tau}^{\gamma,1}+\left(\mathcal{D}_{1}(\mu)(1-\omega)+\mathcal{D}_{0}(\mu)\right)T_{\mu,v+1,\tau}^{\gamma,1}\right\}e_{v}(\omega)\right|+|e_{\tau}(\omega)|$$

$$\leq\left(|\mathcal{D}_{1}(\mu)|+\mathcal{D}_{0}(\mu)\right)\rho^{\mu+\gamma}\sum_{v=0}^{\mu+\gamma-1}T_{\mu,v,\tau}^{\gamma,1}+(\mathcal{D}_{1}(\mu)+\mathcal{D}_{0}(\mu)+|\mathcal{D}_{1}(\mu)|)\rho^{\mu+\gamma}\sum_{v=0}^{\mu+\gamma-1}T_{\mu,v+1,\tau}^{\gamma,1}+\rho^{\tau}$$

$$\leq\left(|\mathcal{D}_{1}(\mu)|+\mathcal{D}_{0}(\mu)\right)\rho^{\tau}\sum_{v=0}^{\mu+\gamma-1}T_{\mu,v,\tau}^{\gamma,1}+(\mathcal{D}_{1}(\mu)+\mathcal{D}_{0}(\mu)+|\mathcal{D}_{1}(\mu)|)\rho^{\tau}\sum_{v=0}^{\mu+\gamma-1}T_{\mu,v+1,\tau}^{\gamma,1}+\rho^{\tau}$$

$$\leq\left\{\left(|\mathcal{D}_{1}(\mu)|+\mathcal{D}_{0}(\mu)\right)\mathcal{B}_{\mu,\gamma}(e_{\tau},1)+(\mathcal{D}_{1}(\mu)+\mathcal{D}_{0}(\mu)+|\mathcal{D}_{1}(\mu)|)\mathcal{B}_{\mu,\gamma}(e_{\tau},1)+1\right\}\rho^{\tau}$$

$$\leq\left\{\left(|\mathcal{D}_{1}(\mu)|+\mathcal{D}_{0}(\mu)\right)\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}+(\mathcal{D}_{1}(\mu)+\mathcal{D}_{0}(\mu)+|\mathcal{D}_{1}(\mu)|)\left(\frac{\mu+\gamma}{\mu}\right)^{\tau}+1\right\}\rho^{\tau}$$

$$= \left\{(1+2|\mathcal{D}_1(\mu)|)\left(1+\frac{\gamma}{\mu}\right)^{\tau}+1\right\}\rho^{\tau}$$

$$\leq 2(1+|\mathcal{D}_1(\mu)|)(1+\gamma)^{\tau}\rho^{\tau}$$

$$\leq 2(1+|\mathcal{D}_1(\mu)|)[\rho(1+\gamma)]^{\tau}(\mu+\gamma-1)$$

$$\leq 2(1+|\mathcal{D}_1(\mu)|)[\rho(1+\gamma)]^{\tau}(\tau-1)\frac{\tau}{\mu+\gamma-1}$$

$$\leq 2(1+\mathcal{N})[\rho(1+\gamma)]^{\tau}(\tau-1)\frac{\tau}{\mu}.$$

Merging Case 1 with Case 2, the wanted outcome is obtained.

ii) As to simultaneous approximation, representing Ω the circle of radius $\rho_1 > \rho$ and center origin, because of $|u-\omega| \geq \rho_1 - \rho$ for any $|\omega| \leq \rho$ and $u \in \Omega$, by Cauchy's formulas it tracks that

$$\left|\left(\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)\right)^{(s)} - \psi^{(s)}(\omega)\right| \leq \frac{s!}{2\pi}\int_{\Omega}\frac{\left|\mathcal{B}^1_{\mu,\gamma}(\psi,u)-\psi(u)\right|}{|u-\omega|^{s+1}}|du|$$

$$\leq \frac{s!}{2\pi}\frac{M^1_{\rho_1,\gamma}(\psi)}{\mu}\frac{2\pi\rho_1}{(\rho_1-\rho)^{s+1}} = \frac{M^1_{\rho_1,\gamma}(\psi)s!\,\rho_1}{\mu(\rho_1-\rho)^{s+1}}.$$

To obtain the equivalence, we establish the qualitative Voronovskaya-type theorem for the operator (10).

Theorem 3.3. Let ρ and $\mathcal{R}$ be constants such that $1 \leq \rho$ and $\mathcal{R} > \gamma + 1$, $\gamma \in \mathbb{N} \cup \{0\}$. Moreover, ψ is analytical in $H_{\mathcal{R}}$, $\mathcal{R} > 1$, with $\psi(\omega) = \sum_{\tau=0}^{\infty} c_\tau \omega^\tau$ and $\mathcal{L}_1 = \lim_{\mu\to\infty} \mathcal{D}_1(\mu)$. Then, for all $\mu \in \mathbb{N}$ we get

$$\lim_{\mu\to\infty}\mu\left[\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)-\psi(\omega)\right] = \left[\gamma\omega + \frac{(1-2\omega)}{2}(1+\mathcal{L}_1)\right]\psi'(\omega) + \frac{\omega(1-\omega)}{2}\psi''(\omega)$$

uniformly in $\overline{H}_\rho$.

Proof. Considering Theorem 2.2, in conformity with the Vitali theorem (see, p. 1 in [2]), it is enough to demonstrate that the sequence

$$\left\{\mu\left[\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)-\psi(\omega)\right]\right\}_{\mu\in\mathbb{N}}$$

of analytical functions in $H_{\mathcal{R}}$ is uniformly bounded in each $\overline{H}_\rho$, $1 \leq \rho < \mathcal{R}$. Indeed, the consequence is immediate outcome of Theorem 3.2 (i) and hence, we obtain

$$\left|\mu\left[\mathcal{B}^1_{\mu,\gamma}(\psi,\omega)-\psi(\omega)\right]\right| \leq M^1_{\rho,\gamma}(\psi),$$

for all $\mu \in \mathbb{N}$ and $\omega \in \overline{H}_\rho$ with $1 \leq \rho \leq \rho(1+\gamma) < \mathcal{R}$. This completes the proof. Therefore, there exist some constants $0 < \mathcal{J}_1, \mathcal{J}_2 < \infty$ such that

$$\frac{\mathcal{J}_1}{\mu} \leq \left\|\mathcal{B}^1_{\mu,\gamma}(\psi)-\psi\right\|_\rho \leq \frac{\mathcal{J}_2}{\mu}$$

on $\overline{H}_\rho$.

Afterwards, using Theorem 3.2 and Theorem 3.3 we find out the next.

Theorem 3.4. In consideration of assumptions of Theorem 3.2;

$i)$ If ψ is not a polynomial of degree 0, for all $1 \leq \rho < \rho(1+\gamma) < \mathcal{R}$, we get

$$\left\|\mathcal{B}^1_{\mu,\gamma}(\psi) - \psi\right\|_\rho \sim \frac{1}{\mu}, \mu \in \mathbb{N},$$

in which the constants in the equivalence depend upon $\psi, \rho, \gamma, \mathcal{N}$ and $\mathcal{L}_1$.

$ii)$ If $1 \leq \rho < \rho_1 < \rho_1(1+\gamma) < \mathcal{R}$ and ψ is not a polynomial of degree $\leq s-1$, $s \in \mathbb{N}$, we get

$$\left\|\left(\mathcal{B}^1_{\mu,\gamma}(\psi)\right)^{(s)} - \psi^{(s)}\right\|_\rho \sim \frac{1}{\mu}, \mu \in \mathbb{N},$$

in which the constants in the equivalence depend upon $\psi, \rho, \rho_1, \gamma, s, \mathcal{N}$ and $\mathcal{L}_1$.

4. CONCLUSION

In the present study, we introduce complex perturbed Bernstein-Schurer operators attached to analytical functions in a disk centered at the origin having radius greater than one and scrutinize some approximation properties. Therefore, we extend approximation results of the newly defined operator to larger sets in the complex plane than the real interval.

Researches on approximation properties for linear operators in various complex domains have some practical applications across a wide range of fields, including signal processing, optimization and machine learning. For example, in signal processing, comprehending the approximation properties of operators in the complex domain is essential for tasks such as signal denoising, compression, reconstruction, leading improving performance in the applications like image and audio processing. By discovering these practical applications, this research presents new tools for applications in real-world scenarios requiring complex functional analysis. In the light of developments in science and technology, future work may emerge from both the theoretical and the applicative perpectives. Future research may contain discovering numerical applications of these operators in real-world scenarios and also generalizing the results to the other types of complex operators in various domains in $\mathbb{C}$.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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